Exact radial closed orbit distortion with E821 β-function

Rushabh Mehta, Eric M. Metodiev, William M. Morse
Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

1 Introduction

In the Muon $g-2$ experiment, muons circulate in a storage ring with electric quadrupoles providing a restoring force to the particles in the ring in the vertical direction. In this technical note, the orbit distortion due to field error will initially be calculated under the assumption that the quadrupoles are located everywhere along the ring, namely in the case of constant β, and then corrections are made to fit the actual experiment, where around 40% of the ring is covered by quadrupoles.

2 Calculations

From Ref. [1], the closed orbit distortion due to field errors at an angle ϕ around the ring can be obtained from the expression:

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = \nu^2\beta^{3/2}F(\phi), \tag{1}$$

where $\Delta x = \sqrt{\beta}\eta$ is the distortion of the closed orbit, ϕ is the angle around the ring, ν is the tune constant, and $F(\phi)$ is the field error which we represent by a Fourier transform:

$$F(\phi) = \sum_{N=0}^{\infty} \frac{B_N}{B_0 R} \cos[N\phi + \phi_N]. \tag{2}$$

Weng and Mane found the solution to this differential equation through the Green function method [1]. The result is expressed by the integral:

$$\eta(\phi) = \frac{\nu}{2\sin(\pi\nu)} \int_{\phi}^{\phi+2\pi} \beta^{3/2}(\phi')F(\phi')\cos[\nu(\pi + \phi - \phi')]d\phi'. \tag{3}$$

2.1 Constant β-function

In the approximation that the quadrupoles cover the entire ring, we have a constant $\beta = R/\nu$. Integrating Eq. 3 using `Mathematica`, we find:

$$\frac{\Delta x}{R} = \sum_{N=0}^{\infty} \frac{\cos[N\phi + \phi_N]}{-N^2 + 1 - n} \frac{B_N}{B_0}. \tag{4}$$

where we have used the fact that $\nu = \sqrt{1-n}$.

2.2 E821 β-function

For non-constant beta functions, direct numerical integration of Eq. 3 is performed in place of analytic evaluation. `Mathematica` was used to perform the numerical integration. The β-function for the E821 ring is taken to be:

$$\beta(\phi) = 7.625\text{m} + 0.275\text{m}\cos[4\phi] + 0.01\text{m}\cos[8\phi] \tag{5}$$

The resulting distortion to the closed orbit is compared to the constant case in Eq. 4. The parameters used in the calculation are:

$$R = 7.112\text{m}$$
$$\frac{B_N}{B_O} \sim 20\text{ppm}$$
$$\nu = 0.93.$$

The following figures shows the distortion to the closed orbit for the constant β case and the E821 beta function case for $N = 1, 2$ and 3 and $\phi_N = 0$. The two distortions are very close, only differing slightly for $N = 1$, but increase rapidly for $N = 2, 3$, with percent differences of 1%, 2%, and 10%.

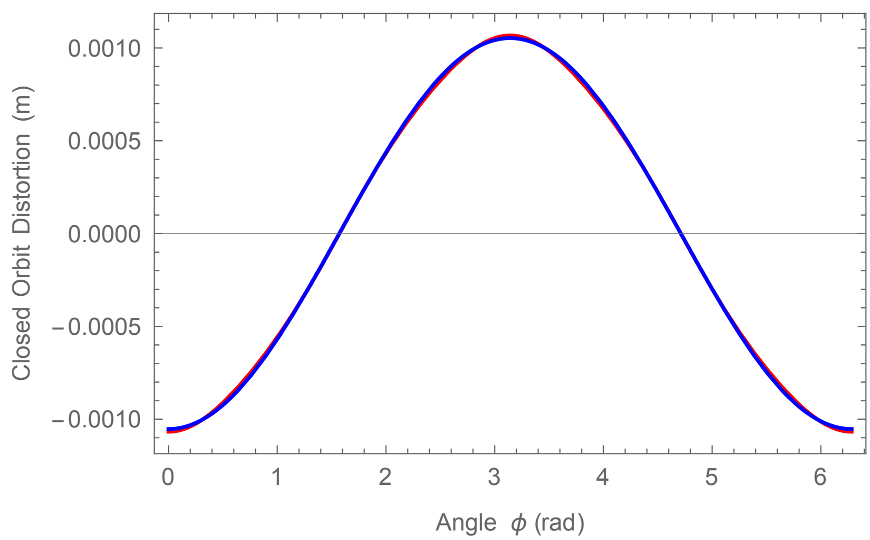

Figure 1: The distortion to the closed orbit in the constant β (blue) and E821 β (red) cases for $N = 1$ and $\phi_N = 0$. The differences are tens of microns.

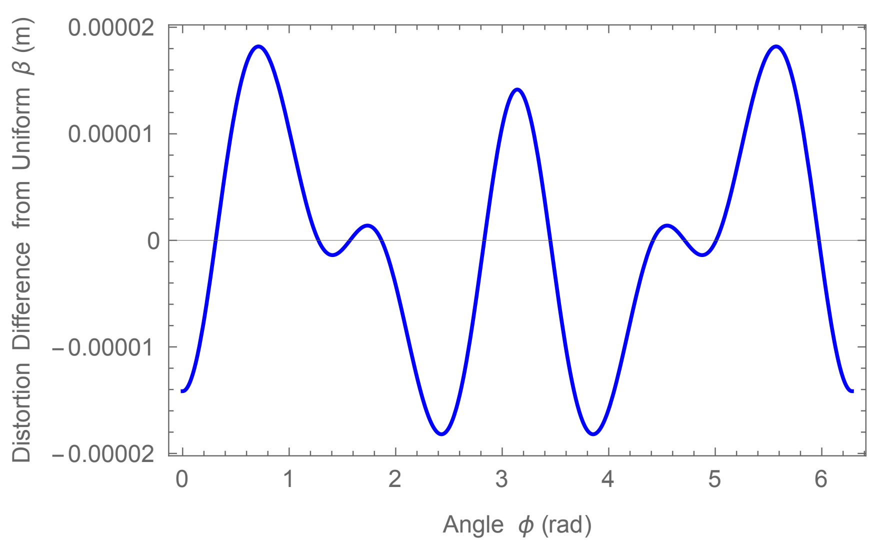

Figure 2: The difference of the closed orbit distortions between the constant β and E821 β cases for $N = 1$ and $\phi_N = 0$.

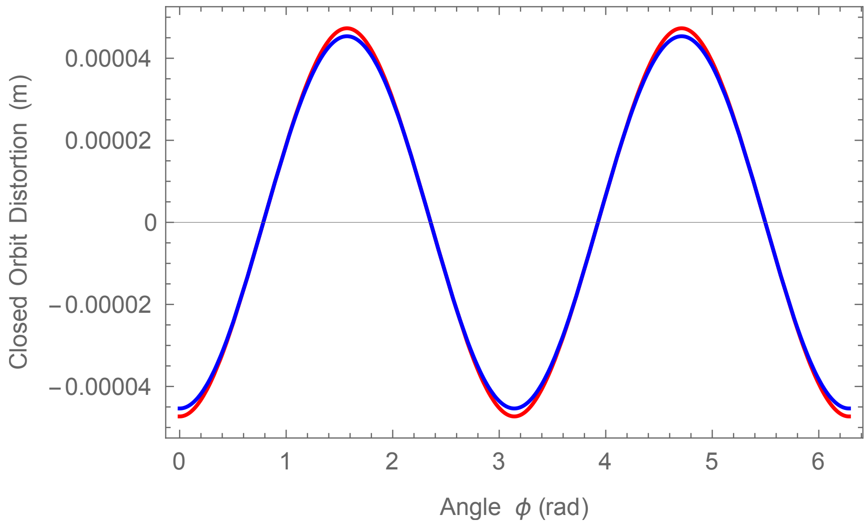

Figure 3: The distortion to the closed orbit in the constant β (blue) and E821 β (red) cases for $N = 2$ and $\phi_N = 0$. The differences are tens of microns.

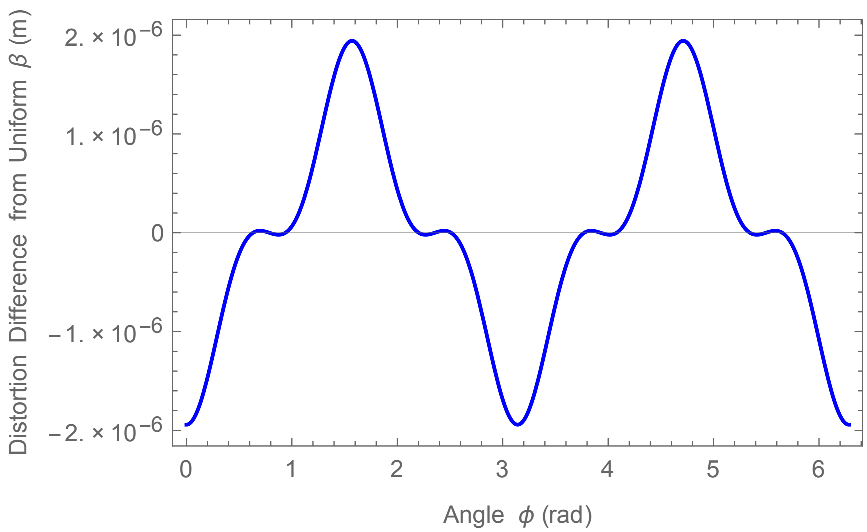

Figure 4: The difference of the closed orbit distortions between the constant β and E821 β cases for $N = 2$ and $\phi_N = 0$.

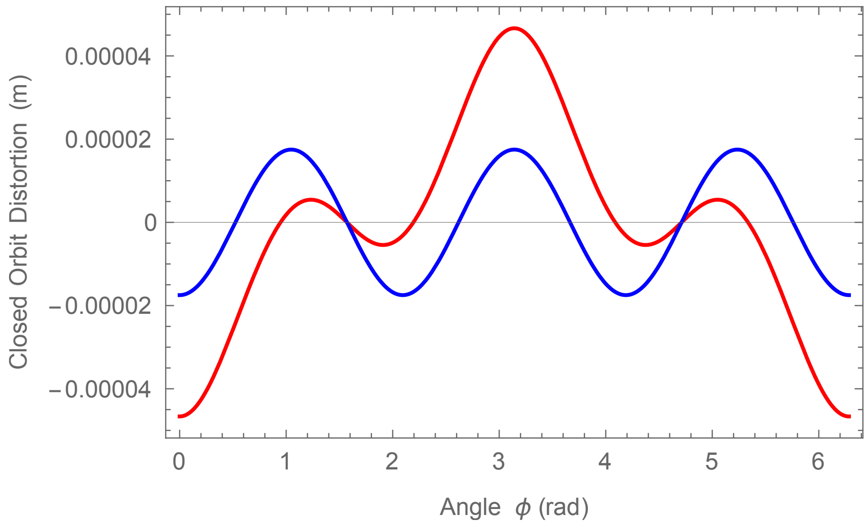

Figure 5: The distortion to the closed orbit in the constant β (blue) and E821 β (red) cases for $N = 3$ and $\phi_N = 0$. The differences are tens of microns.

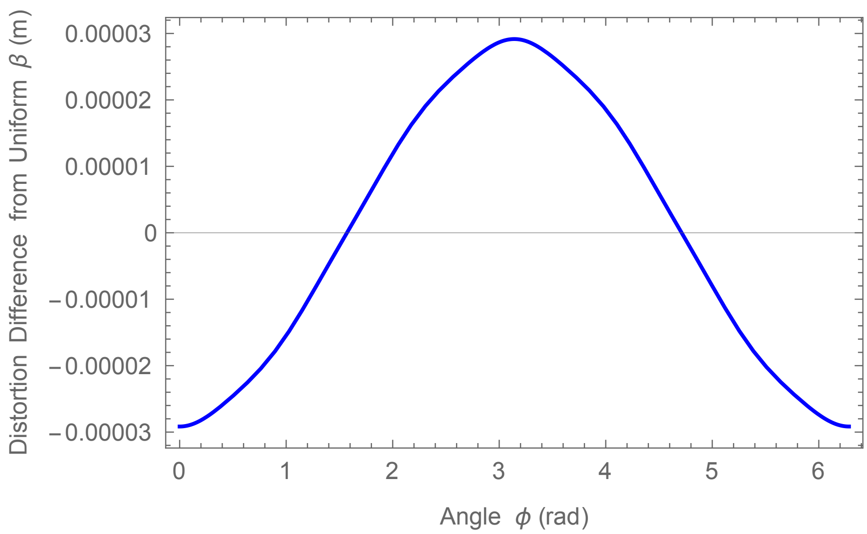

Figure 6: The difference of the closed orbit distortions between the constant β and E821 β cases for $N = 3$ and $\phi_N = 0$.

For a complete study of the difference between the closed orbits in the constant β approximation and with the realistic E821 β function, the difference is calculated for $N \in \{0, 1, 2, 3\}$ for $\phi \in [0, 2\pi]$ and $\phi_N \in [0, 2\pi]$. The resulting heat maps of the differences between the constant and realistic β cases are shown in Figure 7.

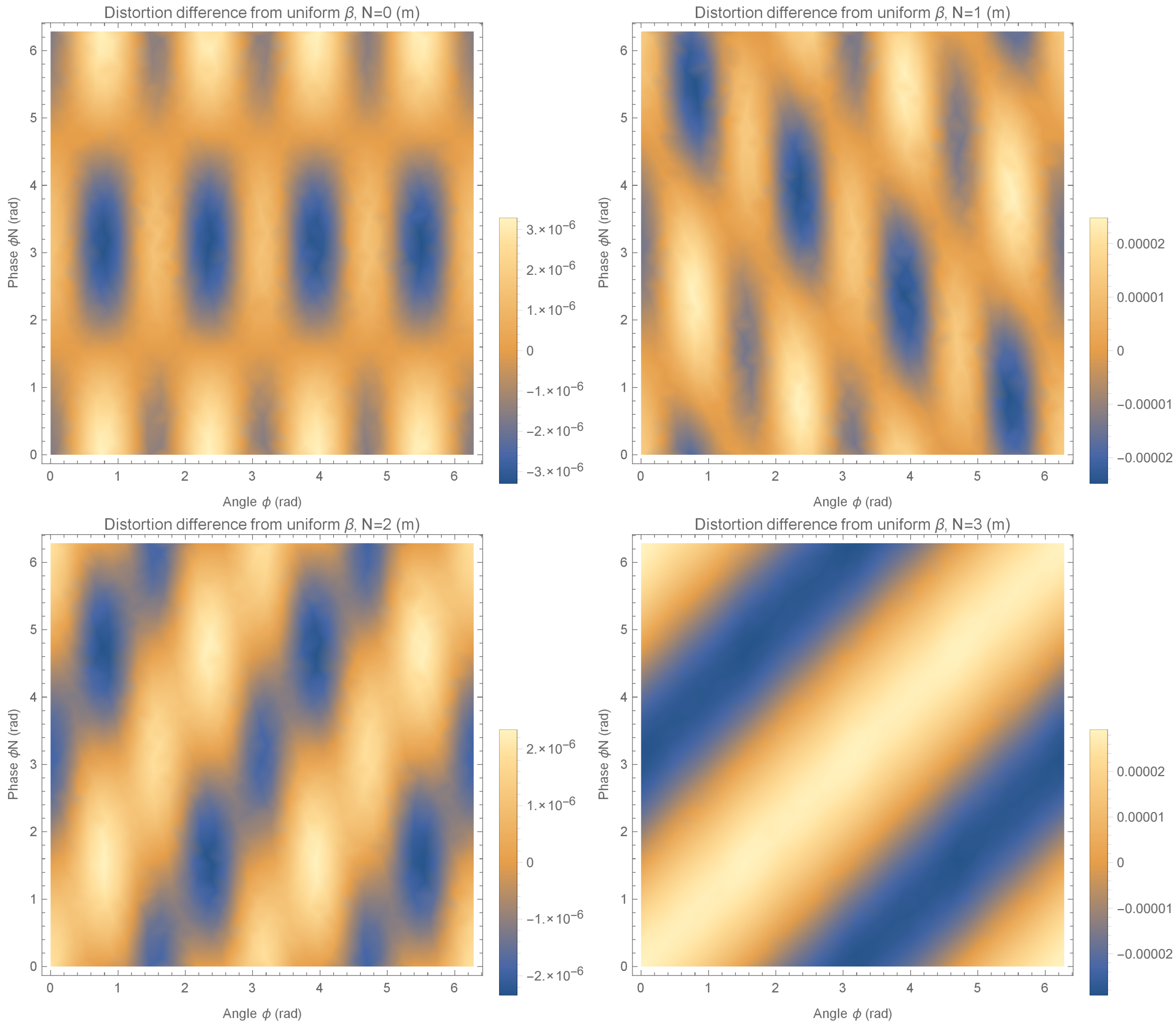

Figure 7: The difference in the predicted closed orbit distortions between the constant β case and the E821 β function case over all ϕ and ϕ_N values. The cases of $N \in \{0, 1, 2, 3\}$ are plotted.

The differences are on the order of 2×10^{-5}m for $N = 1$ and $N = 3$ and on the order of 2×10^{-6}m for $N = 0$ and $N = 2$. The largest deviation from the prediction of the constant β case seems to occur for $N = 5$, as shown in Figure 8 below for $N = 5$ and $\phi_N = 0$.

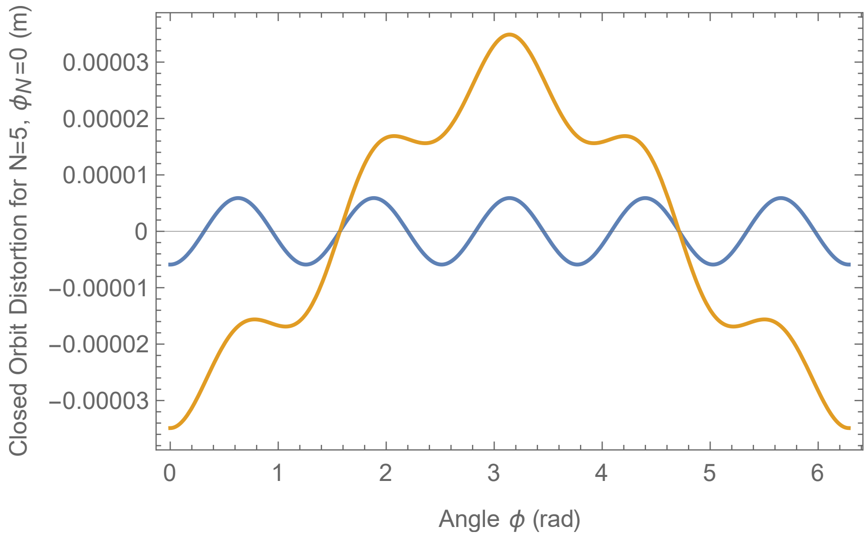

Figure 8: The distortion to the closed orbit in the constant β (blue) and E821 β (yellow) cases for $N = 5$ and $\phi_N = 0$. This case is the largest calculated deviation from the constant β approximation.

3 Discussion

In this note, Eq. 3 is the equation from which the exact closed orbit distortion can be calculated numerically. The results were compared to the constant β case of Eq. 4 over a variety of N values and phases ϕ_N.

From the present analysis, it is clear that the direct numerical integration to determine the distortion of the closed orbit for the realistic E821 β function is non-trivially different than the constant β approximation. For the $N = 1$ case, the difference in the predicted distortions is on the order of 1%. The deviation is larger for $N = 2$ through $N = 6$ and then the two cases begin to agree once again for larger N. The heat maps in Figure 7 present the entire difference over the entire parameter ranges of ϕ_N and ϕ values.

References

[1] W.T. Weng and S.R. Mane. *Fundamentals of Particle Beam Dynamics and Phase Space*. Brookhaven National Laboratory. September 4, 1991.