GALOIS COVERS AND THE FUNDAMENTAL GROUP

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Abstract. This paper is a brief introduction to the Galois Correspondence in topology. First, Galois theory is developed over finite extensions and over covers of topological spaces. Then, using Grothendieck's construction of Galois Theory, a correlation is established between the fundamental group and absolute Galois groups.

Contents

1. Introduction

There is a long tradition of parallels between Galois theory and covering spaces, from Hilbert's ideas on the connections between number fields and Riemann surfaces [1] to Grothendieck's Galois theory, aimed at studying the fundamental group in the setting of algebraic geometry [2]. In this paper, we develop the language and tools necessary to understand Galois Theory and its relationship with the fundamental group. In sections 2 and 3, we develop Galois Theory on finite dimensional extensions, and explore corresponding tools on covers over topological spaces. In sections 4 and 5, we consider more general extensions through Grothendieck's Galois Theory, and use it to show a direct relationship between the absolute Galois group and the fundamental group.

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Some familiarity with groups and fields, including the notions of algebraic and separable extensions/closures as presented in [4] is assumed of the reader. Familiarity with basic concepts of topology as defined in chapter 0 of [5] is also assumed.

Sections 4 and 5 require basic familiarity with categories. Here, chapters 1 and 4 of [6] present a concise summary of the tools used in this paper.

2. Finite Galois Extensions

Given a field extension L/k, we denote by $\mathrm{Aut}(L/k)$ the group of automorphisms of L that fix all elements of k. There is a natural left action of this group on the extension, given by $(\phi, l) \mapsto \phi(l)$.

Since the automorphism group $\mathrm{Aut}(L/k)$ must preserve the field structure of L, the set of all elements fixed by the action must be a field extension of k.

Definition 2.1. An algebraic extension L/k is called a **Galois extension over** k if the field of all elements fixed by $\mathrm{Aut}(L/k)$ is k. We denote $\mathrm{Aut}(L/k)$ as $\mathrm{Gal}(L/k)$ for these extensions.

This notion of a Galois extension has several desirable properties, which we will explore in the following theorems.

Theorem 2.2. *A Galois extension L/k is separable, and for each $l \in L$, the minimal polynomial over k of l splits into linear factors in L.*

Proof. Let $\alpha \in L$, and let $m_\alpha(x) \in k[x]$ be the minimal polynomial of α. Consider f to be the product $\prod(x - \phi(\alpha))$, where $\phi(\alpha)$ ranges over all distinct values it can take with $\phi \in \mathrm{Gal}(L/k)$.

$f \in k[x]$ since there are only a finite number of distinct $\phi(\alpha)$ as each $\phi(\alpha)$ is a root of m_α (we can see this by noticing that $m_\alpha(\phi(\alpha)) = \phi(m_\alpha(\alpha)) = \phi(0) = 0$.)

We also know that since m_α is the minimal polynomial for α, $m_\alpha | f$. But, for all $\phi \in \mathrm{Gal}(L/k)$, we showed that $\phi(\alpha)$ is a root of m_α. As such, $m_\alpha = f$. By construction, f has no multiple roots, is a product of linear factors, and m_α is irreducible. □

The converse of this theorem also happens to be true, but is not necessary for this paper. However, in order to discuss various Galois extensions of a given field, it is convenient to have a maximal Galois extension to embed other Galois extensions in.

Lemma 2.3 ([4])**.** *Let k be a field.*

(1) There exists an algebraic closure $\overline{k}$ of k, unique up to isomorphism.
(2) Given algebraic extension L/k, there exists an embedding $L \to \overline{k}$ that is fixed on k.
(3) Given the extension embedding from (2)*, there exists an isomorphism from $\overline{L} \to \overline{k}$ that is the embedding on the domain L.*

The **separable closure** of k, denoted by k^{sc} or K, the smallest sub-field of the algebraic closure of k that contains all finite separable extensions of k, is both Galois over k, and is maximal (via the inclusion ordering). We show this result formally in Theorem 2.4.

Theorem 2.4. *Given field k, and K the separable closure of k, K is the maximal Galois extension of k.*

Proof. First, we show that K is maximal via inclusion for Galois extensions. Suppose there existed a Galois extension L/k such that there exists an $\alpha \in L$ that is not in K. This implies that α is not separable in k (if it was, $k(\alpha)$ would be a finite separable extension of k.) and hence L/k is not separable. By Theorem 2.2, this is a contradiction.

Next, we show that it is Galois over k. To do this, we must show that for all $\alpha \in K$ that are not in k, there exists $\phi \in \mathrm{Aut}(K/k)$ such that $\phi(\alpha) \neq \alpha$.

Since $\alpha \notin k$, its minimal polynomial m_α must have degree 2 or higher. Since m_α factors into linear factors in K, it has another root in K. Let α' be another root of m_α. Since m_α is irreducible, there exists a natural isomorphism between $k(\alpha)$ and $k(\alpha')$ that fixes k and maps $\alpha \to \alpha'$. By (3) in Lemma 2.3, we can extend this map to $\overline{k} \to \overline{k}$. We conclude by noting that separable elements must map to other separable elements (because of their minimal polynomials) which implies that the above map, limited to K, is in fact the desired ϕ. □

We refer to $\mathrm{Gal}(K/k)$ as the **absolute Galois group of** k. It is worth noting that the separable closure is unique within a given algebraic closure. However, as algebraic closures are unique only up to isomorphism, so are separable closures.

The next theorem provides us a very useful way to prove that extensions are Galois.

Theorem 2.5. *Let L/k be a separable field extension. L/k is Galois if for all $\phi \in Gal(K/k)$, $\phi(L) \subseteq L$.*

Proof. We showed in Theorem 2.4 that for any element $\alpha \in K$ with minimal polynomial m_α over k, and for any $\phi \in \mathrm{Gal}(K/k)$, $m_\alpha(\phi(\alpha)) = 0$. Hence, $\phi(\alpha)$ is a root of m_α regardless of the choice of ϕ. By Theorem 2.2, if $\alpha \in L$, then m_α splits into linear factors in L, implying that all of its roots are in L, as desired.

In the other direction, consider $\alpha \in L$ that is not in k. Since K is Galois over k, there exists $\phi \in \mathrm{Gal}(K/k)$ such that $\phi(\alpha) \neq \alpha$. By assumption, we know that $\phi(\alpha) \in L$, so ϕ limited to L is an automorphism of L that does not fix α. □

With these results, we can finally state the central result of Galois theory for finite extensions, known as the Galois correspondence. This theorem is loosely based on Theorem 10.2 in [3].

Theorem 2.6. *Let L/k be a finite Galois extension, $G = Gal(L/k)$. There exists an inclusion-reversing bijection between subgroups $H \subseteq G$ and sub-field extensions $k \subseteq F \subseteq L$ via the maps*

$$F \to Aut(L/F)$$
$$H \to L^H$$

where L^H is the field of elements fixed by H. Moreover, L/F is Galois, and F/k is Galois if $H \trianglelefteq G$, in which case $Gal(F/k) \cong G/H$.

Proof. First, let F be a field such that $k \subseteq F \subseteq L$. L/F being Galois is a direct consequence of Theorem 2.5. Designating $H = \mathrm{Gal}(L/F)$, we note that $L^H = F$, since the elements fixed by H are exactly F.

Next, let $H \subseteq G$. L is Galois over L^H by virtue of L being a field; moreover, $\mathrm{Gal}(L/L^H) = H$ as only automorphisms in H preserve L^H, and those automorphisms only preserve L^H.

If $H \trianglelefteq G$, consider the action of G/H on $F = L^H$. The natural action is well defined since H fixes F. Since L/k is Galois, we have

$$G/H = \mathrm{Gal}(F/F^{G/H}) = \mathrm{Gal}(F/L^G) = \mathrm{Gal}(F/k)$$

as desired.

Finally, if F/k is Galois, note that every $\phi \in G$ maps F to itself (See Lemma 2.3 and Theorem 2.4), so consider the map $G \to \mathrm{Gal}(F/k)$ which simply restricts the automorphisms to F. The kernel of this map is a subgroup of G that fixes F. If we call the kernel H, we have $H \trianglelefteq G$, and $G/H = \mathrm{Gal}(F/k)$ as desired. $\square$

This result gives us a way of associating groups, which act to permute generators of extensions over base fields, with field extensions in which the generators live. [2] does an excellent job in demonstrating the power that this relationship provides, but these exercises are outside of the scope of this paper. Instead, we are interested in developing a similar tool set for topology, and understanding its implications.

Before we leave the study of finite Galois theory, here are a few examples of Galois and not-Galois extensions, and their corresponding groups.

Example 2.7. Given $n \in \mathbb{N}$, $n > 1$, let ω be a primitive nth root of unity. $\mathbb{Q}(\omega)/\mathbb{Q}$ is Galois since all of roots of the minimal polynomial of ω, the nth cyclotomic polynomial, are in $\mathbb{Q}(\omega)$. (We know roots must map to other roots, so by Theorem 2.5, the extension is Galois). The roots of the cyclotomic polynomial are generated by coprime powers to n of ω, meaning that the Galois group is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.

Example 2.8. Let $a \in \mathbb{Q}$ such that it does not have a cube root in $\mathbb{Q}$. The extension $\mathbb{Q}(\sqrt[3]{a})/\mathbb{Q}$ is not Galois, since all automorphisms of the extension must fix $\sqrt[3]{a} \in \mathbb{R}^+$ (since the other roots of the minimal polynomial $x^3 - a$ are complex.)

Example 2.9. Let $a \in \mathbb{Q}$ have an irrational square root. $\mathbb{Q}(\sqrt{a})/\mathbb{Q}$ is clearly Galois (the extension contains all roots of $x^2 - a$, and $\mathbb{Q}(\sqrt[4]{a})/\mathbb{Q}(\sqrt{a})$ is also Galois (the extension contains all roots of $x^2 - \sqrt{a}$). However, $\mathbb{Q}(\sqrt[4]{a})/\mathbb{Q}$ is not Galois, since there is no automorphism that is not the identity on $\sqrt{a}$.

3. Galois Covers of Topological Spaces

We saw in the last section that Galois theory is the theory of the correspondence between field extensions and the symmetry groups that act on them. We will be able to analyze covers, continuous surjections with locally discrete fibers, in a similar fashion and develop a similar bijection between groups and topological spaces.

3.1. **Covering Spaces and Morphisms.**

Definition 3.1. Given topological spaces X, Y, Y is a cover of X if there exists a continuous map $\rho : Y \to X$ such that for all $x \in X$, there exists open neighborhood $V \subseteq X$ where $\rho^{-1}(V) = \coprod_{i \in I} U_i$ is the disjoint union of open $U_i \subseteq Y$ such that ρ restricted to any of the U_i induces a homeomorphism between U_i and V.

These maps can be more easily seen in the commutative diagram in Figure 1.

Example 3.2. If Y is a cover such that we could choose $V = X$ for every point in X, the cover is known as a **trivial cover**. Given some indexing set I with a discrete topology, Y is homeomorphic to $X \times I$.

$$\begin{tikzcd} \rho^{-1}(V) \arrow[rr] \arrow[dr, "\rho"'] & & V \times I \arrow[dl, "f"] \\ & V & \end{tikzcd}$$

FIGURE 1. Cover Commutative Diagram. For each point in X, we find an open V such that this diagram commutes (and f is the forgetful map ignoring I, the index set)

Example 3.3. Let X, Y both be the unit circle in $\mathbb{C}$ and let $\rho : z \to z^n$ for some $n \in \mathbb{N}$. The indexing set I in this case is $\mathbb{Z}/n\mathbb{Z}$. This map is shown visually in Figure 2.

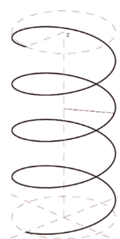

FIGURE 2. Covering map. This graph plots the result of $z \to z^n$ on the $x - y$ plane against $\arg z$ on the z-axis. The map sends a circle to a circle wrapped n times.

Given the notion of covers, we now want to think about morphisms over covers

Definition 3.4. Given covers Y, Z of X via maps ρ_Y, ρ_Z, a **cover morphism** is a continuous function $f : Y \to Z$ that satisfies the commutative diagram in Figure 3.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "\rho_Y"'] & Z \arrow[d, "\rho_Z"] \\ & X \end{tikzcd}$$

FIGURE 3. Cover Morphism Diagram. We want the morphism to respect the covering maps of the covers.

With the notion of a morphism over covers, we also can define $\mathrm{Aut}(Y/X)$ for Y a cover of X.

3.2. **Group Actions on Covers.**

Definition 3.5. Let G be a group acting on topological space Y. Define the topological space $G \setminus Y$ to be the set of orbits of the action, with the finest topology that makes the natural map $Y \to G \setminus Y$ continuous.

We would like group actions to induce covering morphisms. In order to ensure they do so, we need to place a restriction on the types of morphisms we consider.

Definition 3.6. Let G be a topological group, and Y a topological space upon which G acts continuously from the left. The action is **properly discontinuous** if for all $y \in Y$, there is an open set V containing y such that for all $g_1, g_2 \in G$, $g_1 V \cap g_2 V = \varnothing$.

Theorem 3.7. *If G is a group with a properly discontinuous action on connected space Y, the natural map $\rho : Y \to G \setminus Y$ makes Y a cover of $G \setminus Y$.*

Proof. By Definition 3.6, for every $y \in Y$, we can find V such that the action of G on V is pairwise disjoint. Clearly, V is open in $G \setminus Y$, and its preimage in Y is a union of disjoint open sets from the action. The natural map induces a homeomorphism between each open set and V since every element of g has a continuous inverse. So, Y covers $G \setminus Y$. □

This method of constructing covering maps allows us to produce many interesting examples, including the following.

Example 3.8 ([5])**.** $\mathbb{Z}$ acts on $\mathbb{R}$ via translation ($x \to x + z$). This gives us a cover $\mathbb{R}$ over $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $[0, 1) \subset \mathbb{R}$.

Example 3.9. Let G be the group of the nth roots of unity. G acts on $\mathbb{C} \setminus \{0\}$ by multiplication, and the action is properly discontinuous, as for any $re^{i\theta}$ in $\mathbb{C} \setminus \{0\}$, we can consider $V = \{se^{i\phi} \mid s \in \mathbb{R}^+,\ \theta - \frac{i\pi}{n} < \phi < \theta + \frac{i\pi}{n}\}$. Clearly, since V is a open sector with an angle of $\frac{2i\pi}{n}$, and all actions rotate the sector by multiples of $\frac{2i\pi}{n}$, we have a properly discontinuous action.

3.3. Galois Covers.

Let X be our base topological space that is **locally connected** – every neighborhood of a point has a connected open neighborhood of the point as a subset.

Theorem 3.10. *Let $\rho : Y \to X$ be a cover, and Z be a connected space. If $f, g : Z \to Y$ are continuous maps such that $\rho \circ f = \rho \circ g$, and if there exists $z \in Z$ such that $f(z) = g(z)$, then $f = g$.*

Proof. Let V be a connected open set in X that satisfies the cover criterion for $\rho(f(z)) \in X$ (possible since X is locally connected.) Let U be the open set in the disjoint union of $\rho^{-1}(V)$ that contains $f(z)$. Hence, there exists some open neighborhood X of z such that f, g both map X to U. Since ρ^{-1} is continuous on V, we have that

$$\rho^{-1} \circ \rho \circ f = \rho^{-1} \circ \rho \circ g$$
$$f = g$$

on $X \subset Z$. This implies $E = \{z \in Z \mid f(z) = g(z)\}$ is open.

Suppose there exists $z' \in Z$ such that $f(z') \neq g(z')$. Then, by a construction parallel to that of z, we find that there is an open neighborhood $X' \subset Z$ such that $f(X') \cap g(X') = \varnothing$. This implies E is closed.

Since Z is connected and E is non-empty, we have that $E = Z$, so $f = g$ on Z. □

Corollary 3.11. *A non-trivial automorphism of cover $Y \to X$ must have no fixed points.*

With these theorems in hand, we can show that the action of the automorphism group on a space induces a covering map.

Theorem 3.12. *If $\rho : Y \to X$ is connected, then the action of Aut(Y/X) is properly discontinuous.*

Proof. For $y \in Y$, let V be the connected open set in X that contains $\rho(y)$ and such that $\rho^{-1}(V)$ is a disjoint union of open sets. Let U_y be the open set in the union that contains Y. We seek to show that U_y satisfies Definition 3.6. Given two automorphisms $g_1, g_2 \in \mathrm{Aut}(Y/X)$, we know that $g_1 U_y$ and $g_2 U_y$ must map to U_1, U_2, open sets that are part of the disjoint union. Hence, they are disjoint. □

We now begin to see the similarity between automorphism groups of coverings spaces and Galois groups. Galois groups permute roots of polynomials over the base field, while cover automorphisms permute the fibers of the cover. For a cover automorphism group to be Galois, we expect that every fiber is permuted by the group. We will formalize this intuition in Definition 3.13.

Definition 3.13. Let $\rho : Y \to X$ be a connected cover. Consider the maps

$$Y \to \mathrm{Aut}(Y/X) \setminus Y \to X$$

where the second map is induced by ρ. $\mathrm{Aut}(Y/X)$ is Galois if the induced map is a homeomorphism. We denote these automorphism groups $\mathrm{Gal}(Y/X)$.

This definition needs some explanation. We know that the elements of $\mathrm{Aut}(Y/X)\setminus Y$ are the orbits of Y over the group action. However, each fiber must be a union of orbits of the covering map. When the induced map is a homeomorphism, each fiber is its own orbit.

With this intuition, we can formulate the notion of a Galois cover in the following way.

Theorem 3.14. *A connected cover $\rho : Y \to X$ is Galois if and only if Aut(Y/X) acts transitively on each fiber of ρ.*

Proof. When $\mathrm{Aut}(Y/X)$ acts transitively on the fibers of ρ, the orbits of the action are equivalent to the fibers of ρ, which implies $\mathrm{Aut}(Y/X) \setminus Y \to X$ is a homeomorphism. Conversely, if $\mathrm{Aut}(Y/X) \setminus Y \to X$ is a homemorphism, then there is a bijection between the fibers of X and the orbits of $\mathrm{Aut}(Y/X)$, implying that each fiber is permuted by the action. □

Example 3.15. For a properly discontinuous action G on space Y, the cover $Y \to G \setminus Y$ is Galois with Galois group G.

Proof. Each fiber of the covering map are orbits of the group action G. Hence, the set of fibers of Y is just $G \setminus Y$, which clearly is made into an induced homeomorphism with itself by Definition 3.13. The set of automorphisms are those that send elements of Y to other elements of their fiber (which is also their orbit). The automorphisms that do this are the elements of G. □

Now, we can state the equivalent of Theorem 2.6 for covering maps in Theorem 3.17.

Lemma 3.16 ([5])**.** *Given connected cover $\rho : Y \to X$ and continuous map $f : Z \to Y$, if $\rho \circ f : Z \to X$ is a cover, so is $f : Z \to Y$.*

Theorem 3.17. *Let $\rho_Y : Y \to X$ have automorphism group $G = $ Gal(Y/X). For every $H \subset G$, the map induced by ρ_Y, $\rho'_Y : H \setminus Y \to X$ turns $H \setminus Y$ into a cover of X.*

On the contrary, if there exists a cover morphism f over X from $Y \to Z$, then $f : Y \to Z$ is a Galois cover and $Z \cong Gal(Y/Z) \setminus Y$. The maps

$$H \to H \setminus Y$$
$$Z \to Gal(Y/Z)$$

create an inclusion reversing bijection between the subgroups of the automorphism group and the intermediate covers of X. $\rho_Z : Z \to X$ is Galois if $H \trianglelefteq G$, in which case $Gal(Z/X) \cong G/H$.

Proof. By the definition of a cover, we know that for any $x \in X$, we can find V, an open connected neighborhood of x such that $\rho_Y^{-1}(V)$ splits into disjoint open sets in Y, each homeomorphic to V. In other words, $\rho_Y^{-1}(V) \cong V \times I$ for some indexing set I. For any $H \subset G$, H has a left action on I as each automorphism maps a homeomorphic copy of V to another homeomorphic copy of V. Because of this, $\rho_Y'^{-1}(V) \subset H \setminus Y \cong V \times H \setminus I$, which shows that $H \setminus Y$ is a cover of X.

Lemma 3.16 shows that for Z a cover of X that satisfies the commutative diagram in Figure 3, $f : Y \to Z$ is a cover. To show that the cover is Galois, we will show the condition presented in Theorem 3.3 holds for this cover. Let $z \in Z$, and consider $y_1, y_2 \in f^{-1}(Z)$ (not necessarily distinct). Hence, $y_1, y_2 \in \rho_Y^{-1}(\rho_Z(z))$. As $\rho_Y : Y \to X$ is Galois, there exists a ϕ such that $\phi(y_1) = y_2$. We want $\phi \in \mathrm{Aut}(Y/Z)$, which is equivalent to ϕ preserving the cover map f. So, we want to show that $f(\phi(y)) = f(y)$ for all $y \in Y$. But, by Corollary 3.11, this is true. Hence, $f : Y \to Z$ is Galois and so, $Z \cong \mathrm{Gal}(Y/Z) \setminus Y$.

These results show the inclusion reversing nature of the intermediate covers and subgroups of the Galois group. We conclude by handling the case when $H \trianglelefteq G$.

If $H \trianglelefteq G$, then consider the induced action of G/H on $Z = H \setminus Y$. As the action is induced from G's action on Y, and $\rho_Y = \rho_Z \circ f$, the induced action must preserve ρ_Z. So, we have a group homomorphism from $G/H \to \mathrm{Aut}(Z/X)$ which is injective by definition of Z. However, it is surjective since $(G/H) \setminus Z \cong G \setminus Y \cong X$, so $G/H \cong \mathrm{Aut}(Z/X)$, which proves that $\rho_Z : Z \to X$ is Galois.

If $\rho_Z : Z \to X$ is Galois, then consider $\phi_Y \in G$. We want a homomorphism from $G \to \mathrm{Gal}(Z/X)$ so that the kernel of the homomorphism will be our H. Since $\rho_Z : Z \to X$ is Galois, for any $y \in Y$, there exists map ϕ_Z such that $\phi_Z(f(y)) = f(\phi_Y(y))$ (as $y, \phi_Y(y)$ are in the same fiber of ρ_Y). By Theorem 3.10, we know that ϕ_Z is unique (if there were two maps ϕ_1, ϕ_2 with the property above, consider $\phi_1 \circ \phi_2^{-1}$ with the Theorem). Hence, we have a homomorphism as desired, and the kernel of the map, H, is normal in G. □

4. Grothendieck's Categorical Galois Theory

So far, we have seen an equivalence between the notions of Galois extensions and Galois covers. One of Grothendieck's key insights was to view some base field as a point, and finite separable field extensions over the base field as finite sets of points which all map onto the base point [7]. Then, the absolute Galois group of the base field has a natural action on the set of points, permuting them but keeping the base point fixed.

In this section, we will be able to explicitly show this relationship via the language of categories and functors. Replacing the inclusion reversing bijection will be a contravariant functor, establishing an anti-equivalence between the category of finite sets with a continuous and transitive action from the absolute Galois group, and the category of finite separable sub-extensions of the separable closure.

4.1. **Krull Topology.**

We begin by expanding our notion of the Galois correspondence to more general field extensions. Here, we borrow some definitions and theorems from Krull summarized in [8].

Definition 4.1. Let E/F be a Galois extension. We define the topology of $\mathrm{Gal}(E/F)$ to be generated by subgroups of the form $\mathrm{Gal}(E/K)$, where K/F is a finite separable sub-extension of E/F. This topology is known as the **Krull Topology**.

With the addition of the Krull topology, we can prove some basic relationships between subgroups of the Galois group and extensions.

Theorem 4.2. *For any subextension K/F of E/F, $Gal(E/K)$ is closed in $Gal(E/F)$.*

Proof. Consider $\phi \in \mathrm{Gal}(E/F) \setminus \mathrm{Gal}(E/K)$. We seek to show that there is an open neighborhood of ϕ which is disjoint from $\mathrm{Gal}(E/K)$ to show that the complement of $\mathrm{Gal}(E/K)$ is open. Since $\phi \notin \mathrm{Gal}(E/K)$, there exists some $a \in K$ such that $\phi(a) \neq a$. Consider $\mathrm{Gal}(E/F(a))$. The group is open in the Krull Topology, and it has the property that for any $f \in \mathrm{Gal}(E/F(a))$, $\phi(f(a)) = \phi(a) \neq a$. This implies that $\phi \cdot \mathrm{Gal}(E/F(a)) \cap \mathrm{Gal}(E/K) = \varnothing$, so we are done. $\square$

This theorem suggests that for any subextension of a Galois extension, the Galois group fixing the subextension is closed in the Krull Topology.

We now show the converse to this theorem.

Theorem 4.3. *For any closed subgroup $H \subseteq Gal(E/F)$, $H = Gal(E/E^H)$.*

Proof. $H \subseteq \mathrm{Aut}(E/E^H)$ by definition of fixed fields. We also know that $G \setminus H$ is an open set in G. Since it is open, for any $\phi \in G \setminus H$, there exists some finite extension K/F such that $\phi \cdot \mathrm{Gal}(E/K) \subseteq G \setminus H$ (since one can always find an open neighborhood around a point in an open set). Let K' be the Galois closure of K over F (which is still finite). So, $\phi \cdot \mathrm{Gal}(E/K') \subseteq G \setminus H$ as $\mathrm{Gal}(E/K') \subseteq \mathrm{Gal}(E/K)$, implying that $\phi \notin H \cdot \mathrm{Gal}(E/K')$.

Let $\phi_{K'}$ be ϕ limited to the domain K'. Since $\mathrm{Gal}(E/K')$ is trivial in this domain, if we let $H_{K'}$ be the automorphisms of H restricted to K', we have $\phi_{K'} \notin H_{K'}$. So, there is $a \in E^{H_{K'}} \subset E^H$ such that $\phi_{K'}(a) = \phi(a) \neq a$. This shows that $H = \mathrm{Aut}(E/E^H)$, which also implies that E/E^H is Galois. $\square$

These two theorems give us the impression that the closed subgroups of $\mathrm{Gal}(E/F)$ and the intermediate extensions of E/F are in bijection just like in the finite case. In fact, these theorems are crucial in the proof of the next lemma, a generalization of Theorem 2.6. (The proof is not provided in the text due to its length and similarity to Theorem 2.6.)

Lemma 4.4 ([8])**.** *Let K be a subextension of Galois extension E/F. Then $Gal(E/K)$ is a closed subgroup of $Gal(E/F)$. Moreover, the maps*

$$K \to Aut(E/K)$$
$$H \to E^H$$

create an inclusion-reversing bijection between the subextensions of E over F and the closed subgroups of $Gal(E/F)$. K/F is Galois if $Aut(K/F) \trianglelefteq Gal(E/F)$, in which case $Gal(K/F) \cong Gal(E/F)/Gal(E/K)$.

4.2. **The Functor of Points.**

Let k be our base field, K the separable closure of k. Let F be a finite separable extension of k. We know through our work in Section 2 that the number of embeddings of F into K preserving k is finite. This implies that $\mathrm{Hom}_k(F,K)$ is finite. $\mathrm{Gal}(K/k)$ has a natural left action on $\mathrm{Hom}_k(F,K)$, given by $(g,\phi) \to g\circ\phi$, where $g \in \mathrm{Gal}(K/k),\ \phi \in \mathrm{Hom}_k(F,K)$.

Theorem 4.5. *The left action above of Gal(K/k) on Hom$_k(F,K)$ is continuous and transitive, and Hom$_k(F,K)$ is isomorphic to the left coset space of an open subgroup of Gal(K/k). F/k is Galois if and only if the open subgroup is normal.*

Proof. We know that any finite separable extension can be generated by a single element. Let that element by $\alpha \in F$ with minimal polynomial m_α. For any $\phi \in \mathrm{Hom}_k(F,K)$, α must be sent to another root of m_α. Since $\mathrm{Gal}(K/k)$ permutes the roots of m_α transitively, the action of $\mathrm{Gal}(K/k)$ on $\mathrm{Hom}_k(F,K)$ is transitive.

On the other hand, given $\phi \in \mathrm{Hom}_k(F,K)$, let $S \subseteq \mathrm{Gal}(K/k)$ be the stabilizer of ϕ. S must fix $\phi(F)$ in order to stabilize the morphism, and as such, by Lemma 4.4, S is open in $\mathrm{Gal}(K/k)$. This implies that the action is continuous.

Since the action is transitive and continuous, we have that the map $g\circ\phi \mapsto gS$ is an isomorphism from $\mathrm{Hom}_k(F,K) \to S\setminus \mathrm{Gal}(K/k)$. By Lemma 4.4, $S \trianglelefteq \mathrm{Gal}(K/k)$ if and only if F/k is Galois. □

Now, we want to establish a categorical anti-equivalence between those finite sets that have a continuous and transitive action from the absolute Galois group (like $\mathrm{Hom}_k(F,K)$ from Theorem 4.6) and sub-extensions of the separable closure. We establish this equivalence in Theorem 4.7.

Lemma 4.6 ([6])**.** *Two categories C_1, C_2 are equivalent if there exists a functor $F : C_1 \to C_2$ such that the functor is*

1. *Fully Faithful - The map of sets Hom$(A,B) \to$ Hom$(F(A),F(B))$ is bijective for all $A,B \in\ C_1$.*
2. *Essentially Surjective - Every object of C_2 is isomorphic to an object of the form $F(A)$ for some A.*

If L, M are finite separable extensions of k, then any homomorphism $\phi : L \to M$ respecting k induces a map from $\mathrm{Hom}_k(M,K) \to \mathrm{Hom}_k(L,K)$ by composing ϕ with the homology map. This is a contravariant functor known as the Grothendieck functor, or as the **functor of points**.

Theorem 4.7. *(*[7]*) The contravariant functor mapping the finite separable extension L/k to the finite set Hom$_k(L,K)$ gives an anti-equivalence between the category of finite separable extensions and the category of finite sets with a continuous and transitive left Gal(K/k) action. Galois extensions give finite sets isomorphic to a quotient of Gal(K/k).*

Proof. We need to show that the functor of points of form $\mathrm{Hom}_k(,K)$ satisfies the conditions of Lemma 4.6.

To show that the functor is fully faithful, we need to show that given L, M finite separable extensions over k, the map from $L \to M$ to $\mathrm{Hom}_k(M,K) \to \mathrm{Hom}_k(L,K)$ is bijective. Note that $\mathrm{Gal}(K/k)$ is transitive over both $\mathrm{Hom}_k(M,K),\ \mathrm{Hom}_k(L,K)$. Let $\phi : L \to M$ induce $\phi' : \mathrm{Hom}_k(M,K) \to \mathrm{Hom}_k(L,K)$ as discussed before this

Theorem. Let $\rho \in \mathrm{Hom}_k(M, K)$. Hence by the transitivity, we know that any stabilizer of ρ is also a stabilizer of the induced map $\rho \circ f \in \mathrm{Hom}_k(L, K)$. This implies that $\rho \circ f(L) \subseteq \rho(M)$.

We know that ϕ' commutes with ρ, which shows that $\rho^{-1} \circ \phi' \circ \rho \cong \phi$, but the previous paragraph showed that $\rho^{-1} \circ \phi' \circ \rho$ is in fact the unique map that induces ϕ.

To show that the functor is essentially surjective, we show that any finite set S acted on by $\mathrm{Gal}(K/k)$ is isomorphic to some $\mathrm{Hom}_k(L, K)$. We do so by picking a point $s \in S$, and considering the stabilizer U of s. We know that the action is continuous, which implies that U is open and the fixed field of U is a finite separable extension L/k. The isomorphism we define from $\mathrm{Hom}_k(L, K) \to S$ is $g \circ e \mapsto gs$, where e is the identity embedding, and g is in $\mathrm{Gal}(K/k)$. By definition, the stabilizer of e is U, which implies that both sets are isomorphic to $U \setminus \mathrm{Gal}(K/k)$.

The last part of the Theorem comes directly from Theorem 4.5. □

Theorem 4.7 uses the functor of points to show an anti-equivalence between finite separable extensions of k (which are sub-extensions of its absolute Galois group), and finite sets which have a left action from the absolute Galois group that is transitive (permuting all of the points) and continuous under the Krull Topology.

5. The Fundamental Group and Universal Covers

In the Section 4, we developed a more general interpretation of the Galois correspondence in terms of an anti-equivalence between the categories of sub-extensions of the absolute Galois group of some field and of finite sets acted on continuously and transitively by the absolute Galois group.

In this Section, we will combine our results from all of the previous section to find a similar anti-equivalence between the categories of connected covering spaces of a well-behaved base space and of sets acted on by a group transitively. This, too, will be done using a functor of points, with the role of the absolute Galois group and the separable closure played by the fundamental group and universal cover, respectively.

5.1. The Fundamental Group.

Definition 5.1. A **path** is a continuous map from $[0, 1]$ to space X.

Definition 5.2. A **loop** is a path f where $f(0) = f(1)$.

Definition 5.3. Two paths $f, g : [0, 1] \to X$ are **homotopic** if there exists a continuous map $h : [0, 1]^2 \to X$ such that $h(0, x) = f(x),\ h(1, x) = g(x)$.

These definitions are due to [5].

Theorem 5.4. *Homotopies of paths are an equivalence relation.*

Proof. We check the three necessary conditions.

(1) Reflexivity: $h(x, y) = f(x) \to f(x) \sim f(x)$.
(2) Symmetry: If h is the homotopy between f, g, define $h'(x, y) = h(1 - x, y)$. This is a homotopy between g, f.
(3) Transitivity: If m, n are the homotopies between f, g and g, h, then

$$p(x, y) = \begin{cases} m(2x, y) & x < \frac{1}{2} \\ n(2x - 1, y) & x \geq \frac{1}{2} \end{cases}$$

is a homotopy between f, h.

$\square$

Definition 5.5. Given paths $f, g : [0,1] \to X$ such that $f(0) = g(1)$, the composition of paths

$$f \circ g(x) = \begin{cases} g(2x) & x < \frac{1}{2} \\ f(2x-1) & x \geq \frac{1}{2} \end{cases}$$

In particular, one can compose any two loops with the same starting point.

Definition 5.6. The **fundamental group** of a space X around a point $x \in X$, denoted by $\pi_1(X, x)$ or simply $\pi(X, x)$, is the set of equivalence classes (under homotopy) of closed loops $[f]$ with $f(0) = f(1) = x$, equipped with the binary operation of composition.

Theorem 5.7. *$\pi(Z, z)$ is a group.*

Proof. We check the necessary conditions.

(1) Well-defined: If h_1 is a homotopy between f_1, f_2 and h_2 between g_1, g_2, we have

$$h(x, y) = \begin{cases} h_1(2x, y) & x < \frac{1}{2} \\ h_2(2x-1, y) & x \geq \frac{1}{2} \end{cases}$$

as a homotopy between $g_1 \circ f_1$ and $g_2 \circ f_2$.

(2) Identity: $\text{id}(x) = z$ is the identity. Particularly, for any loop $g \in \pi(Z, z)$,

$$h(x, y) = \begin{cases} z & y < \frac{1}{2}x \\ g(\frac{2y-x}{2-x}) & y \geq \frac{1}{2}x \end{cases}$$

is a homotopy from $g \to g \circ \text{id}$. Similarly, we can show that there is a homotopy from $g \to \text{id} \circ g$, showing the property of the identity.

(3) Inverse: The inverse of $[g]$ is $[g']$ (where $g(x) = g'(1-x)$) since there is a homotopy h between $g \circ g^{-1}$ and f, the identity, as follows:

$$h(x, y) = g \circ g^{-1}(2xy)$$

(4) Associativity: The following homotopy h maps from $(g_1 \circ g_2) \circ g_3 \to g_1 \circ (g_2 \circ g_3)$:

$$h(x, y) = \begin{cases} g_1(\frac{4y}{1+x}) & y < \frac{1+x}{4} \\ g_2(4y - 1 - x) & \frac{1+x}{4} \leq y < \frac{2+x}{4} \\ g_3(2y - 1 + 2x(y-1)) & \frac{2+x}{4} \leq y \end{cases}$$

$\square$

Example 5.8. The fundamental group of a connected graph $G = (V, E)$ is the free group of $|E| - |V| + 1$ elements, since the spanning tree of the graph has $|V| - 1$ edges, and edge contracting each of these edges doesn't change the fundamental group. The remaining graph is a single node with all of the remaining edges as loops of the node.

The fundamental group is defined in terms of a base point, but for spaces that are **path-connected** (every two points have a path connecting them), $\pi(Z, x) \cong \pi(Z, y)$ for all $x, y \in Z$. For these spaces, the fundamental group is sometimes denoted as $\pi(Z)$.

Definition 5.9. A path connected space X is **simply connected** if $\pi(X)$ is trivial.

Remark 5.10. Given topological spaces X, Y with $x \in X$, and some continuous map $\psi : X \to Y$, there is a natural map $\pi(X, x) \to \pi(Y, \psi(x))$ given by $\phi \mapsto \psi(\phi)$. This map respects the homotopy equivalence classes of the fundamental group, and is a homomorphism since it respects loop composition.

This suggests the existence of a functor from a category of topology spaces to the category of groups. However, we need to be careful here, since topological spaces are not equipped with a base point, and as such, the above map can not be directly seen as a functor. There are two standard resolutions to this.

Certain texts, such as [9], choose to define **pointed spaces**, topological spaces with a base point, and define morphisms between pointed spaces as morphisms of the underlying spaces that send the base point to the other base point. We then can define a functor from $(X, x) \to \pi(X, x)$, and define the morphism lift from $\mathrm{Hom}((X, x), (Y, y)) \to \mathrm{Hom}(\pi(X, x), \pi(Y, y))$ as above.

Other texts, such as [10], choose to still consider the category of topological spaces, but instead of considering the fundamental group, they consider the fundamental groupoid, a set with homotopy equivalence classes of all loops in a given space, equipped with the binary operation of composition if the two loops have a common base point. As such, the functor maps the category of topological spaces to a category of groupoids.

In this paper, we will stay closer to the approach presented in [9] and deal with pointed spaces. As such, we will now require that covering maps and morphisms send base points to other base points. Pointed spaces will either be denoted as (X, x) to represent $x \in X$ as the base point of X, or simply as X if the base point is not of relevance.

5.2. **The Monodromy Action.**

We want to show that the fundamental group of a base pointed space (X, x) has a left action on fibers of a pointed covering space. In order to do so, we first show that given a cover $\rho : (Y, y) \to (X, x)$, the fundamental group $\pi(X, x)$ has a left action on the fiber $\rho^{-1}(x)$.

Theorem 5.11. *Let $\rho : (Y, y) \to (X, x)$ be a cover. Given path $f : [0, 1] \to X$ with $f(0) = x$, there exists a unique $f' : [0, 1] \to Y$ such that $f'(0) = y$ and $\rho \circ f' = f$.*

Proof. Uniqueness is directly implied by Theorem 3.10 (let X, Y, Z in the Theorem be $X, [0, 1], Y$.) In the case of a trivial cover, existence is obvious (let $y = (x, i) \in X \times I$, then $f'(x) = (f(x), i)$).

For a non-trivial cover, for each $t \in f([0, 1])$, choose an open neighborhood V_t that satisfies the definition of a cover. $(f^{-1}(V_t))_{t \in f([0,1])}$ is hence an open cover of $[0, 1]$, so by compactness, pick a finite subcover. Since the subcover is finite, one can pick $\{t_1, ..., t_n\} \in [0, 1]$ such that each interval $[t_i, t_{i+1}]$ is fully contained in one of the open sets covering $[0, 1]$. Hence, the cover is trivial over each $f([t_i, t_{i+1}])$. Then, induct over i by finding the unique f' with $f(t_i)$ equal to the ending point of the previous path. Composing these f's piece-wise will produce the desired map. □

Corollary 5.12 ([5]). *Let $\rho : Y \to X$ be a cover, and let $f, g : [0, 1] \to X$ be homotopic. There exists a unique $g' : [0, 1] \to Y$ such that $g'(0) = f'(0)$ and $\rho \circ g' = g$ (where f' is defined in Theorem 5.11). Moreover, $g'(1) = f'(1)$.*

Theorem 5.11 and its corollary are known as the lifting lemmas, and ensure that the following construction of a fundamental group action on the fibers of a cover is well defined.

Construction 5.13. Given cover $\rho : Y \to X$ and a point $x \in X$, we can construct an action of $\pi(X, x)$ on $\rho^{-1}(x)$. Let $y \in \rho^{-1}(x)$, $\phi \in \pi(X, x)$. Let $f' : [0, 1] \to Y$ be the lifted path defined in Theorem 5.11 of ϕ with $f'(0) = y$. Our group action is

$$\begin{aligned} \pi(X, x) \times \rho^{-1}(x) &\to \rho^{-1}(x) \\ (\phi, y) &\mapsto f'(1) \end{aligned}$$

Corollary 5.12 ensure that the representative for the homotopy class of $\pi(X, x)$ does not affect the action, making the action well-defined. This action is known as the **monodromy action** [9].

Note that the monodromy action of the fundamental group mirrors the action of the absolute Galois group on homomorphisms in Theorem 4.7.

With the monodromy action at hand, we can define our functor of points, which we will call FIBER_x (x being the base point of our connected and locally simply connected space X). FIBER_x sends a cover $\rho : (Y, y) \to (X, x)$ to $\rho^{-1}(x)$. Construction 5.13 defines the left action of $\pi(X, x)$ on $\rho^{-1}(x)$ (our set of points), and sends a cover morphism $f : Y \to Z$ to a map from the unique lift from X to Y of a loop going through $x \in X$ starting at some $y \in Y$ to the unique lift of the same loop from X to Z starting at $f(y) \in Z$. (The unique lifts are guaranteed by Theorem 5.11). This indeed defines a functor since cover morphisms respect the fibers of covering maps.

5.3. **The Universal Cover.**

Note the similarities between the monodromy action and the action of the absolute Galois group from Section 4. This similarity suggests a similar construction of functor between a category of covers and sets acted on by the fundamental group. In order to establish such an equivalence, we would like to have an object to embed covers in, a topological analogue to the separable closure.

Definition 5.14. The **universal cover** Y of a connected topological space X is a simply connected space that is a cover of X.

Ideally, we would like to embed all of the connected covers of a given space in its universal cover. However, in order to guarantee the existence of the universal cover, we need a few properties to ensure that the base space is well-behaved [5].

Definition 5.15. A space X is **locally path connected** if for any $x \in X$, there exists an open neighborhood U of x such that it is path-connected.

Definition 5.16. A space X is **locally simply connected** if any open set containing point $x \in X$ contains a open subset also containing $x \in X$ that is simply connected.

From this point onwards, we assume that our base space X is connected and locally simply connected. Under these conditions, we can construct a universal cover of X.

Remark 5.17. Local simply connectedness is in fact not a necessary condition for the existence of a universal cover, as semi-local simply connectedness along with local path connectedness also guarantees its existence. See [5] for a construction of the cover with this assumption.

Construction 5.18. Let $\widetilde{X}_x$ be a space whose underlying set is the set of homotopy classes of paths of X starting at $x \in X$.

We define $\rho : (\widetilde{X}_x, \mathrm{id}) \to (X, x)$ as follows. Let $[y] \in \widetilde{X}_x$ ($[y]$ is the homotopy class of y). Let $\rho([y]) = y(1)$. Note that this map is well-defined on the homotopy classes of $\widetilde{X}_x$ since homotopic maps share endpoints. Moreover, since X is locally path-connected, the subset of X that is mapped to by ρ is open, as is its complement in X. Since X is connected, ρ must be surjective.

We will define the topology on this set by defining the basis of open neighborhoods of a given point $[\widetilde{y}] \in \widetilde{X}_x$. Let $y = \widetilde{y}(1)$. Since X is locally simply connected, let U be a simply connected open neighborhood of y. Finally, we take some path $z : [0,1] \to X$ such that $z(0) = y$ and $z([0,1]) \subseteq U$, and compose it with $\widetilde{y}$. The set of all such compositions under homotopy equivalence classes forms the desired open set. For a given $[\widetilde{y}]$, the basis of the open neighborhoods is generated by forming an open set as above for each possible choice of an open simply connected neighborhood of y.

We know that this basis is independent of the representative of $[\widetilde{y}]$ chosen, since the endpoints of all representatives must be the same, ensuring that the choices of the simply connected open neighborhoods and the paths within them will also my the same. This ensures that the topology is well-defined.

We now show that this construction is, in fact, a universal cover of X.

First, we show that the topology is in fact, a topology. Here, we refer to the properties of a local basis of open neighborhoods [9], and find that the only non-trivial condition that we must show is that given any $\widetilde{U}, \widetilde{V}$ in the basis of open neighborhoods of $\widetilde{y} \in \widetilde{X}_x$, there exists some $\widetilde{W}$ in the basis such that $\widetilde{W} \subseteq \widetilde{U} \cap \widetilde{V}$. However, we note that for both $\widetilde{U}, \widetilde{V}$, there are corresponding open simply connected sets U, V in X that contain y. So, $U \cap V$ is open and also contains y, so since X is locally simply connected, there exists $W \subseteq U \cap V$ open neighborhood containing y. Let $\widetilde{W}$ be the open set in the basis of $\widetilde{X}_x$ around $[\widetilde{y}]$ that is generated by choosing W as the simply connected open set. Then, by construction, we have that $\widetilde{W} \subseteq \widetilde{U} \cap \widetilde{V}$. This shows that we have defined a topology.

Next, we show that ρ is continuous. Consider open set U in X that contains $y \in X$. $\forall w \in X$, let $U_w \subseteq U$ be a simply connected open set containing w (possible by local simply connectedness). Note that by construction, $\rho^{-1}(U_w)$ is open in $\widetilde{X}_x$. Finally, we know that $\rho^{-1}(U) = \bigcup\limits_{w \in U} \rho^{-1}(U_w)$, and we know that arbitrary union of open sets is open.

We now show that ρ is a covering map. Let $y \in X$, and U be a simply connected open set containing Y. Note that for any $[\widetilde{y}] \in \rho^{-1}(y)$, and $z \in U$, since U is path-connected, there exists path g such that $g(0) = y,\ g(1) = z,\ g([0,1]) \subseteq U$. So, $\rho([\widetilde{y} \circ g]) = z$. g is unique up to homotopy since U is simply connected. So, for a given $[\widetilde{y}] \in \rho^{-1}(y)$, there exists an open set in $\widetilde{X}_x$ that is mapped homemorphically onto U via ρ. So all we must show is that the sets generated by each element of $\rho^{-1}(y)$ are disjoint. However, this is true simply because given distinct classes $[\widetilde{y}], [\widetilde{y}_2] \in \rho^{-1}(y)$, and some g with $g(0) = y,\ g([0,1]) \subseteq U$, $\widetilde{y} \circ g \not\simeq \widetilde{y}_2 \circ g$ since

$\widetilde{y} \not\cong \widetilde{y}_2$ and U is simply connected in X. This shows that ρ is a cover, since for any $y \in X$, there is a simply connected set containing it in X.

Finally, we show that $\widetilde{X}_x$ is simply connected. It is path connected since any path $\gamma : [0,1] \to X$ has a path to the constant path via the map $f : [0,1] \times [0,1] \to X,\ f(t_1, t_2) = \gamma((1-t_1) \cdot t_2)$. This map is continuous by construction of $\widetilde{X}_x$'s topology. Given any loop in $\widetilde{X}_x$, we can likewise apply the map above to send it continuously to the constant path, implying that all loops are homotopic, and the fundamental group is trivial.

Example 5.19. The universal cover of the unit circle (embedded in $\mathbb{C}$) is $\mathbb{R}$ via the covering $z \mapsto e^{iz}$. The automorphism group of the cover is $\mathbb{Z}$ (we can imagine bending $\mathbb{R}$ into an infinite helix, which then is projected onto the unit circle).

In the next few theorems/constructions, we will prove properties of the universal cover that allow other covers to be embedded in it.

Construction 5.20. Let $\psi : Y \to X$ be a cover with X having base point x. For all $y \in \psi^{-1}(x)$, we define the covering morphism $f_y : \widetilde{X}_x \to Y$ as follows.

We let f_y map $[\phi] \in \widetilde{X}_x$ to $\widetilde{\phi}(1)$, where $\widetilde{\phi}$ is the unique lift of the path ϕ to Y with $\widetilde{\phi}(0) = y$ via Theorem 5.11. Corollary 5.12 ensures that f_y is well defined on the homotopy classes of $\widetilde{X}_x$, and it is a cover morphism since $\psi(\widetilde{\phi}) = \psi(y) = x$.

Consider the case of $Y = \widetilde{X}_x$. In this case, the covering morphisms are endomorphisms over $\widetilde{X}_x$, and there is a natural bijection induced by the construction above between $\psi^{-1}(x)$ and $\mathrm{Aut}(\widetilde{X}_x/X)$. The identity is one of these morphisms, so we define the **universal element** of X to be the element $e \in \widetilde{X}_x$ such that $f_e = \mathrm{id}$ as constructed above.

Theorem 5.21. *Given space (X, x) with universal cover $\rho : \widetilde{X}_x \to X$, $\textsc{Fiber}_x$ is represented by ρ – $\textsc{Fiber}_x$ is isomorphic to $Hom_X(\rho,)$.*

Proof. We seek to find a natural isomorphism between the two functors. Concretely, for some cover $\psi : Y \to X$, every point y of $\psi^{-1}(x)$ should correspond to a cover morphism $f_y : \widetilde{X}_x \to Y$. Construction 5.20 gives us a map from $\psi^{-1}(x) \to \mathrm{Hom}_X(\widetilde{X}_x, Y)$. By construction, this map is injective. Conversely, given some cover morphism $g : \widetilde{X}_x \to Y$, notice that $g(e)$, the image of the universal element of X, is a member of $\psi^{-1}(x)$, and that $g = f_{g(e)}$ as defined in Construction 5.20. This defines a left inverse of the map, and it naturally also is injective. This shows us that the original map is a bijection. To show that this map is a natural isomorphism, we simply need to show that it is a natural transformation. The cover morphism $Y \to Z$ over X that maps some $y \in Y$ to some $z \in Z$ induces the map $\mathrm{Hom}_X(\widetilde{X}_x, Y) \to \mathrm{Hom}_X(\widetilde{X}_x, Z)$ by sending $f_y \to f_z$, since f_y sends $e \to y$ and f_z sends $e \to z$. □

Theorem 5.21 is crucial since it, along with Lemma 3.16, implies that for any cover of X, $\widetilde{X}_x$ is a cover of it with a surjective covering map. Like we did with the separable closure in Section 4, we will be able to use this embedding to consider the action of the group $\mathrm{Aut}(\widetilde{X}_x/X)$ on covers of X. We will associate this action with the monodromy action later in this Section.

Finally, we want to show that, like the separable closure, the universal cover is Galois.

Lemma 5.22. *([5]) Covers of simply connected and locally path connected spaces are trivial.*

Lemma 5.23. *([5]) Let X be locally simply connected. If Y, Z are spaces such that $\rho_1 : Y \to X$, $\rho_2 : Z \to Y$ are both covers, $\rho_1 \circ \rho_2 : Z \to X$ is also a cover.*

Theorem 5.24. *$\rho : \widetilde{X}_x \to X$ is Galois.*

Proof. By Theorem 3.3, we simply need to show that $\mathrm{Aut}(\widetilde{X}_x/X)$ is transitive over $\rho^{-1}(x)$. For each $y \in \rho^{-1}(x)$, Theorem 5.21 tells us that there is an endomorphism $\psi_y : \widetilde{X}_x \to \widetilde{X}_x$ that sends the universal element e to y. If $\psi_y \in \mathrm{Aut}(\widetilde{X}_x/X)$, we are done.

Since $\widetilde{X}_x$ is connected, by Lemma 3.16 we have that ψ_y is a covering map for $\widetilde{X}_x$ over itself. Consider some $z \in \psi_y^{-1}(e)$. By Lemma 5.23, we know that $\rho \circ \psi_y$ is also a self cover of $\widetilde{X}_x$, so we can apply Theorem 5.21 to get a map $\psi_z : \widetilde{X}_x \to \widetilde{X}_x$ with $\psi_z(e) = z$ and $\rho \circ \psi_y \circ \psi_z = \rho$. That implies that $\psi_y \circ \psi_z(e) = e$, so by Corollary 3.11, this map is the identity on $\widetilde{X}_x$. As ψ_y, ψ_z are surjective by Theorem 5.21, ψ_y is also injective, so we are done. □

5.4. **The Galois Correspondence of the Fundamental Group.**

Now that we have our topological analogues of the separable closure and absolute Galois group, we can establish a link between the Galois correspondence on covers in Theorem 3.17 and the fundamental group monodromy action via the functor of points, an analogue of Theorem 4.7.

Let (X, x) be a connected and locally simply connected pointed space and let $\rho : Y \to X$ be a cover of X. Consider first the functor $\mathrm{F{\small IBER}}_x$, which sends Y of X to $\mathrm{F{\small IBER}}_x(Y) = \rho^{-1}(x)$, a fiber of the covering map. $\rho^{-1}(x)$ is imparted with a left action from the fundamental group as the monodromy action.

The functor $\mathrm{Hom}_X(\widetilde{X}_x,)$ sends Y to the set of covering morphisms $\mathrm{Hom}_X(\widetilde{X}_x, Y)$. Given some $\phi \in \mathrm{Gal}(\widetilde{X}_x/X)$, and some $\psi \in \mathrm{Hom}_X(\widetilde{X}_x, Y)$, we note that $\psi \circ \phi \in \mathrm{Hom}_X(\widetilde{X}_x, Y)$, and it is not hard to see that this forms a right action of $\mathrm{Gal}(\widetilde{X}_x/X)$ on $\mathrm{Hom}_X(\widetilde{X}_x, Y)$.

Theorem 5.21 showed us that $\mathrm{F{\small IBER}}_x(Y) \cong \mathrm{Hom}_X(\widetilde{X}_x, Y)$, leading us attempt to establish some relationship between the groups acting on each of these sets.

In order to do so, we want both actions to act from the same direction. Hence, we introduce the following notation.

Definition 5.25. Given some group G, let G^{op} be the underlying set of G equipped with the operation $x \cdot y = yx$.

So, $\mathrm{Gal}(\widetilde{X}_x/X)^{op}$ has a left action on $\mathrm{Hom}_X(\widetilde{X}_x, Y)$, which prompts the next theorem.

Theorem 5.26. *Given space (X, x), $Gal(\widetilde{X}_x/X)^{op} \cong \pi(X, x)$*

Proof. Let $[\alpha] \in \pi(X, x)$. Note that for any $[y] \in \widetilde{X}_x$, $[y \circ \alpha] \in \widetilde{X}_x$. Note that this defines a right action of $\pi(X, x)$ on $\widetilde{X}_x$, as composition respects homotopy classes, and is a continuous action. Let ϕ_α be an endomorphism of $\widetilde{X}_x$ that applies the right action of $[\alpha]$ onto $\widetilde{X}_x$. Note that $\phi_\alpha \in \mathrm{Gal}(\widetilde{X}_x/X)^{op}$ since the composition of paths respects the covering map. So, define the following map

$$\pi(X,x) \to \mathrm{Gal}(\widetilde{X}_x/X)$$
$$[\alpha] \mapsto \phi_\alpha$$

This map is a group homomorphism since composition of paths is associative. It is injective since for any $[\alpha] \neq [\mathrm{id}] \in \pi(X,x)$, ϕ_α sends the constant path to α, which by assumption is not homotopic with the constant path, implying that the kernel is trivial.

Let $\phi \in \mathrm{Gal}(\widetilde{X}_x/X)^{op}$, and $[y] \in \widetilde{X}_x$. We know that $\phi(y)(1) = y(1)$ since automorphisms preserve the covering map. Hence, $\alpha = [y^{-1} \circ \phi(y)] \in \pi(X,x)$ and $y \circ y^{-1} \circ \phi(y) = \phi(y)$. So, $\phi^{-1} \circ \phi_\alpha$ fixes $[y]$, so since $\widetilde{X}_x$ is connected, so by Corollary 3.11, $\phi^{-1} \circ \phi_\alpha$ is the identity. Hence, $\phi = \phi_\alpha$, and hence, the homomorphism is surjective. So, we have an isomorphism. □

This Theorem, along with Theorem 3.17, shows that the automorphism group of the universal cover is isomorphic to the fundamental group, and that there is a bijection between subgroups of the fundamental group, and covers of the base field where the automorphism groups of the covers are precisely those subgroups.

The next theorem is a summary of these insights.

Theorem 5.27. *The cover $\rho : \widetilde{X}_x \to X$ is a connected Galois cover with automorphism group isomorphic to $\pi(X,x)$. For any cover $Y \to X$, the left action of Gal$(\widetilde{X}_x/X)^{op}$ on $\textsc{Fiber}_x(Y)$ is the monodromy action.*

Proof. The first part of this Theorem is just a restatement of other theorems in this section. By Theorem 5.21, every point $y \in \rho^{-1}(x)$ induced a covering morphism $f_y : \widetilde{X}_x \to Y$, and that a path $[g] \in \widetilde{X}_x$ is mapped to $g'(1)$, where g' is the lift of g to the space Y with $g'(0) = y$ (via Theorem 5.11). In particular, $f_y(\mathrm{id})$ is the constant path at y. In Theorem 5.26, we showed that $[\alpha] \in \pi(X,x)$ acts on $\mathrm{id} \in \widetilde{X}_x$ by sending it to $[\alpha]$. So, the action of α on the fiber $\rho^{-1}(x)$ will map y to $\alpha'(1)$, where α' is the lift (via Theorem 5.11) of α to Y where $\alpha'(0) = y$. This is the monodromy action. □

What this shows is not only that the automorphism group is isomorphic to the fundamental group, but their actions also are identical.

Our final theorem establishes the Galois correspondence between covers of (X,x) and sets with $\pi(X,x)$ actions. $\textsc{Fiber}_x$ is our functor of points: it maps covers to sets with points that all map to the base point.

Theorem 5.28. *Let (X,x) be a connected and locally simply connected topological space. $\textsc{Fiber}_x$ induces an equivalence between the category of connected covers of X and sets with left transitive $\pi(X,x)$ actions. Sets with transitive $\pi(X,x)$ actions correspond to connected covers, and Galois covers to coset spaces of normal subgroups.*

Proof. As in Theorem 4.7, we seek to satisfy the conditions of 4.6 with our covariant functor $\textsc{Fiber}_x$.

To show that $\textsc{Fiber}_x$ is fully faithful, consider connected covers $\rho_Y : Y \to X$, $\rho_Z : Z \to X$. We want to show that for each map $\phi : \textsc{Fiber}_x(Y) \to \textsc{Fiber}_x(Z)$ of $\pi(X,x)$ sets, there is exactly one $f : Y \to Z$ respecting the commutative diagram in Figure 3 that is sent to ϕ. Let $y \in \textsc{Fiber}_x(Y) = \rho_Y^{-1}(x)$. From Construction

5.20, we have a map $g_y : \widetilde{X}_x \to Y$ that sends $[\mathrm{id}] \mapsto y$. We can now apply Theorem 3.17 to show that $\mathrm{Gal}(\widetilde{X}_x/Y)$ stabilizes y, and that $Y \cong \mathrm{Gal}(\widetilde{X}_x/Y) \setminus \widetilde{X}_x$. Let $\psi_Y : Y \to \mathrm{Gal}(\widetilde{X}_x/Y) \setminus \widetilde{X}_x$.

We also know that $\phi(\mathrm{Gal}(\widetilde{X}_x/Y))$ is a subset of the stabilizer of $\phi(y)$, so hence, if we consider the map $g_{\phi(y)} : \widetilde{X}_x \to Z$ that sends $[\mathrm{id}] \mapsto \phi(y)$, we can let $\psi_Z : \mathrm{Gal}(\widetilde{X}_x/Y) \setminus \widetilde{X}_x \to Z$ be the induced map. Hence, $\psi_Z \circ \psi_Y$ is the unique map from $Y \to Z$.

To show that $\textsc{Fiber}_x$ is essentially surjective, we must show that any set S with transitive left action from $\pi(X, x)$ is isomorphic to a fiber of some cover of X. Let $s \in S$, and U_s be its stabilizer in $\mathrm{Gal}(\widetilde{X}_x/X)$. Let $\rho : U_s \setminus \widetilde{X}_x \to X$ be the induced covering map from the universal cover. Note that the space $U_s \setminus \widetilde{X}_x$ is connected as it is the quotient space of a connected space by a stabilizing set. And, $\rho^{-1}(x)$ is isomorphic to S by construction. □

Note the parallels to Theorem 4.7. The fundamental group and universal cover functioned analogously in this Theorem to how the absolute Galois group and the separable closure functioned in Section 4. We conclude with an example.

Example 5.29. In Example 5.19, we noted that the fundamental group of the universal cover $\mathbb{R}$ of the unit circle is $\mathbb{Z}$, which we visualized by "twisting" $\mathbb{R}$ into an infinite helix. Note that subgroups of $\mathbb{Z}$ are of the form $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$, and they correspond to finite helices with n turns (see Example 3.3).

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